Optimum Aperture Radius for a Gaussian Profile

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We derive the optimum aperture size for photometry of sources whose Point Response Function (PRF) profiles can be approximated as Gaussian. By optimum, we mean the aperture size (radius R_{max}) which maximizes the signal-to-noise ratio. If you're after the answer right away, here it is:

$$R_{\max} \approx 1.5852\sigma \approx 0.6731 FWHM,$$

where σ and FWHM (Full Width Half Maximum) characterize the Gaussian profile's width.

A Gaussian is a good approximation for ground based observations where the cores of PRFs can be modeled by a Gaussian 'seeing' profile. For spaced-based detectors, the profiles are primarily determined by the diffraction pattern formed by the telescope entrance pupil and other elements in the optical path. The simplest of these is an Airy pattern with some central obscuration. It would be interesting to repeat the exercise below for an Airy beam. For now, we stick with the humble Gaussian.

Let the (background-subtracted) counts "signal" from a source integrated out to some radius R from its centroid be:

$$C(R) = \int_0^R g(r) 2\pi r \mathrm{d}r, \qquad (1)$$

where $g(r)$ is a *well sampled* radial profile function (i.e., the PRF which is ~ the PSF in the limit of infinite sampling assuming pixels with uniform responsivity).

The noise in the measurement aperture can be generalized into two components: Poisson noise from the source counts alone with variance $C(R)$, and a "background" component which encompasses all other extraneous noise sources affecting the pixel measurements (e.g., read-noise, other instrumental noise, sky photon noise etc). Let the noise variance (*per pixel*) in this extraneous noise be v. The total variance in the measurement radius out to some radius R, ignoring pixel-to-pixel correlations can be written:

$$\sigma_{tot}^2(R) = C(R) + \pi R^2 v, \qquad (2)$$

where the πR^2 is effectively the number pixels in the aperture given R in pixel units.

The signal-to-noise ratio S/N, as a function of radius can be written:

$$S/N = \frac{C(R)}{\sqrt{C(R) + \pi R^2 v}}$$

$$= \frac{\int_0^R g(r) 2\pi r \mathrm{d}r}{\sqrt{\left[\int_0^R g(r) 2\pi r \mathrm{d}r\right] + \pi R^2 v}} \tag{3}$$

We assume a 2D Gaussian for the source profile:

$$g(R) = \frac{1}{2\pi\sigma^2} e^{-r^2/2\sigma^2}, \tag{4}$$

where σ characterizes the Gaussian's width, and for simplicity we have normalized to unity. Any other normalization (or multiplicative) factor could have been used to return a 'real' count > 1, but this can be factored out and it doesn't affect our computation of the optimum radius.

Substituting (4) in (3) and carrying out the integrations (by parts), yields:

$$S/N = \frac{1 - \exp[-R^2/2\sigma^2]}{\sqrt{1 - \exp[-R^2/2\sigma^2] + \pi R^2 v}}. \tag{5}$$

The figure below shows *S/N* as a function of *R* for different values of the "extraneous" variance per pixel *v*. As expected, when there is no additional pixel variance ($v = 0$), the maximum *S/N* is achieved at $R = \infty$. For any other significant value of *v*, the *S/N* appears to be maximized at $R \approx 1.6\sigma$. For very large *v*, it will converge to a finite result. See below.

The optimum radius can be derived more formally from Eq. (5) by solving for *R* such that:

$$\frac{\partial(S/N)}{\partial R} = 0,$$

and taking the limit as $v \to \infty$. This can be easily done numerically in Mathematica or Maple. The optimum radius (at which *S/N* is maximized) comes out to be $\approx 1.5852\sigma$. Given that FWHM $\approx 2.3548\sigma$ for a Gaussian, we also have: $R_{max} \approx 0.6731$FWHM.

It's also important to note that the *S/N* is fairly insensitive to radius *near* this optimal radius. Deviations from the optimal radius by as much as ±50% generally make little difference. In practice, you may want to use a slightly larger aperture than the optimal. This is because centroiding errors will be more critical for smaller apertures than for larger ones. An aperture radius of $R \approx$ FWHM is a good compromise. Only practice makes perfect!

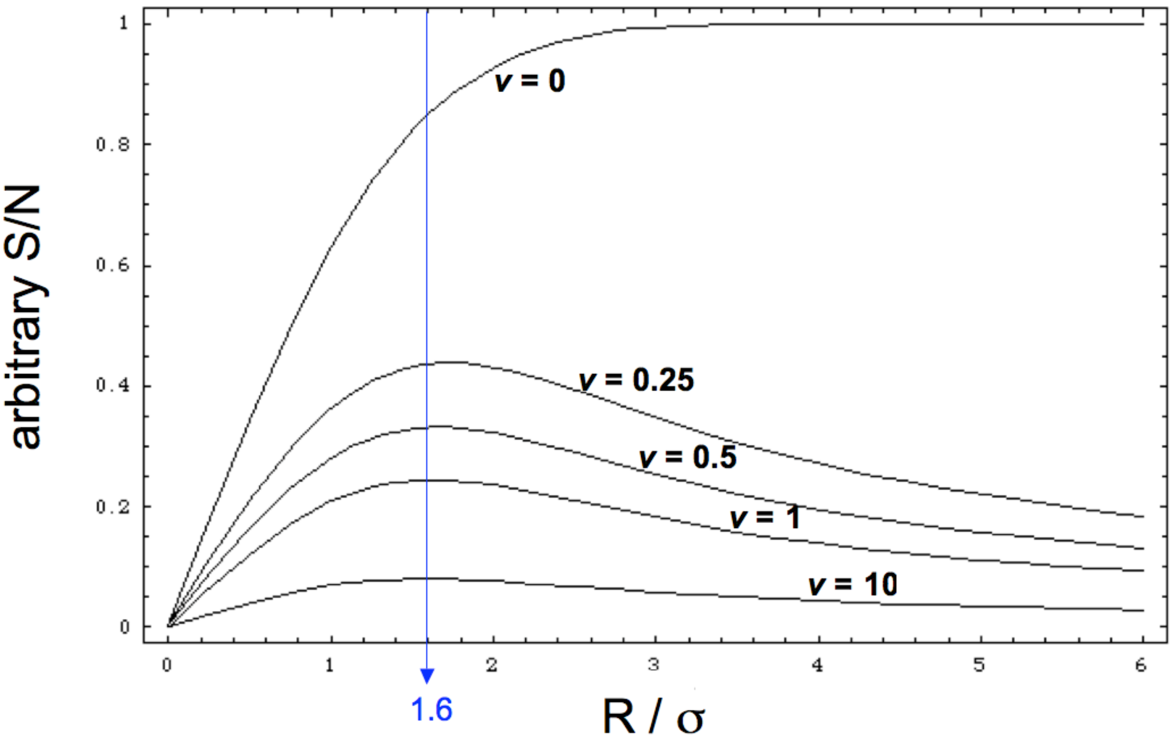

Figure 1: Signal-to-Noise ratio as a function of aperture radius for different values of the "extraneous" pixel noise variance *v* (i.e., all noise components other than the Poisson contribution from the background-subtracted source counts).