Refining scales from linear terms of a distortion fit

Given rotation residual $8\theta$ (after pmg+twist refinement), any relative frame coordinate $(u,v) \rightarrow$ intermediate sky coord $(x,y)$, assuming no distortion can be transformed:

$$
\begin{pmatrix}
  x \\
  y
\end{pmatrix} =
\begin{pmatrix}
  s_1 \cos(8\theta) & -s_2 \sin(8\theta) \\
  s_1 \sin(8\theta) & s_2 \cos(8\theta)
\end{pmatrix}
\begin{pmatrix}
  u \\
  v
\end{pmatrix}
$$

(M)

$$
x = s_1 \cos(8\theta) \cdot u - s_2 \sin(8\theta) \cdot v
$$

$$
y = s_1 \sin(8\theta) \cdot u + s_2 \cos(8\theta) \cdot v
$$

If scales $s_1, s_2$ are in error, i.e. $s_1' + \Delta s_1, s_2' + \Delta s_2$, these errors will map onto the sky as errors:

$$
\Delta x = \Delta s_1 \cos(8\theta) \cdot u - \Delta s_2 \sin(8\theta) \cdot v
$$

(1)

$$
\Delta y = \Delta s_1 \sin(8\theta) \cdot u + \Delta s_2 \cos(8\theta) \cdot v
$$

(2)

Furthermore, if frame coords are distorted by an amount $(\Delta u, \Delta v)$, then (M) also predicts equivalent distortions in the sky projection:

$$
\Delta x = s_1 \cos(8\theta) \cdot \Delta u - s_2 \sin(8\theta) \cdot \Delta v
$$

(3)

$$
\Delta y = s_1 \sin(8\theta) \cdot \Delta u + s_2 \cos(8\theta) \cdot \Delta v
$$

(4)

The linear terms of the distortion polynomials $f(u,v)$ and $g(u,v)$ assuming zero skew are (in zero-index notation):

$$
\Delta u = A_{10} \cdot u + A_{01} \cdot v
$$

(5)

$$
\Delta v = B_{10} \cdot u + B_{01} \cdot v
$$

(6)

We substitute (5) & (6) into (3) & (4) to yield equivalent linear corrections in the sky plane:
\[\Delta x = \left[ S_1 \cos(\theta) \cdot A_{10} - S_2 \sin(\theta) \cdot B_{10} \right] \cdot U + \left[ S_1 \cos(\theta) \cdot A_{01} - S_2 \sin(\theta) \cdot B_{01} \right] \cdot V \] (7)

\[\Delta y = \left[ S_1 \sin(\theta) \cdot A_{10} + S_2 \cos(\theta) \cdot B_{10} \right] \cdot U + \left[ S_1 \sin(\theta) \cdot A_{01} + S_2 \cos(\theta) \cdot B_{01} \right] \cdot V \] (8)

Now equate the linear coefficients in (7) and (8) with those in (1) and (2) to yield:

(1) \Rightarrow
\[\Delta S_1 \cos(\theta) = S_1 \cos(\theta) \cdot A_{10} - S_2 \sin(\theta) \cdot B_{10} \] (9)

- \[\Delta S_2 \sin(\theta) = S_1 \cos(\theta) \cdot A_{01} - S_2 \sin(\theta) \cdot B_{01} \] (10)

(2) \Rightarrow
\[\Delta S_1 \sin(\theta) = S_1 \sin(\theta) \cdot A_{10} + S_2 \cos(\theta) \cdot B_{10} \] (11)

\[\Delta S_2 \cos(\theta) = S_1 \sin(\theta) \cdot A_{01} + S_2 \cos(\theta) \cdot B_{01} \] (12)

Take square of (9) and add it to the square of (11) to yield:

\[\Delta S_1^2 = (S_1 A_{10})^2 + (S_2 B_{10})^2 \] (13)

Similarly, square (10) and (12) and add together:

\[\Delta S_2^2 = (S_1 A_{01})^2 + (S_2 B_{01})^2 \] (14)

By construction, the change in the scales are defined as:
\[ \Delta s_1 = S_{\text{new}} - S_{\text{old}} \]

\[ \Delta s_2 = S_{2\text{new}} - S_{2\text{old}} \]

and all the \( s_1, s_2 \) appearing in the RHS of eqns (13), (14) pertain to the original (old) scales. (13) \& (14) can be re-written:

\[
S_{\text{new}} = S_{\text{old}} + \sqrt{(S_{\text{old}} \cdot A_{\text{old}})^2 + (S_{\text{old}} \cdot B_{\text{old}})^2} \tag{15}
\]

\[
S_{2\text{new}} = S_{2\text{old}} + \sqrt{(S_{2\text{old}} \cdot B_{\text{old}})^2 + (S_{\text{old}} \cdot A_{\text{old}})^2} \tag{16}
\]

If there were no residual rotation bias (and no skew!), the cross terms: \( A_{\text{old}} = 0 \) and \( B_{\text{old}} = 0 \) and (15), (16) reduce to the familiar refinement relations:

\[
S_{\text{new}} = (1 + A_{\text{old}}) S_{\text{old}}
\]

\[
S_{2\text{new}} = (1 + B_{\text{old}}) S_{2\text{old}}
\]

To estimate the rotational bias error \( (8\theta) \), we divide eqn (9) by \( \cos(\theta) \) to give:

\[
\Delta s_1 = S_1 \cdot A_{\text{old}} - S_2 \cdot \tan(8\theta) \cdot B_{\text{old}} \tag{17}
\]

re-arranging (17), we get:
\[
\tan(\delta \Theta) = \frac{S_1 A_{10} - \Delta S_1}{S_2 B_{10}}
\] (18).

We substitute eqn. (13) into (18):

\[
\Rightarrow \tan(\delta \Theta) = \frac{S_1 A_{10} - \sqrt{(S_1 A_{10})^2 + (S_2 B_{10})^2}}{S_2 B_{10}}
\]

With a little algebra, we get:

\[
\tan(\delta \Theta) = \frac{-x}{1 + N \sqrt{1 + x^2}}
\] (19).

where \( x = \frac{S_2 B_{10}}{S_1 A_{10}} \)

If the cross-term coefficient \( B_{10} = 0 \), then it means that \( \delta \Theta = 0 \), as expected.