# Refining scales from linear terms of a distortion fit

Given rotation residual $\delta\theta$ (after ptg + twist refinement), any relative frame coordinate $(u,v) \rightarrow$ intermediate sky coord $(x,y)$; assuming no distortion can be transformed:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} S_1 \cos(\delta\theta) & -S_2 \sin(\delta\theta) \\ S_1 \sin(\delta\theta) & S_2 \cos(\delta\theta) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \qquad (M)$$

$$\Rightarrow \quad x = S_1 \cos(\delta\theta).u - S_2 \sin(\delta\theta).v$$
$$y = S_1 \sin(\delta\theta).u + S_2 \cos(\delta\theta).v$$

If scales $S_1, S_2$ are in error, i.e. $S_1' + \Delta S_1$, $S_2' + \Delta S_2$, these errors will map onto the sky as errors:

$$\Delta x = \Delta S_1 \cos(\delta\theta).u - \Delta S_2 \sin(\delta\theta).v \qquad (1)$$
$$\Delta y = \Delta S_1 \sin(\delta\theta).u + \Delta S_2 \cos(\delta\theta).v \qquad (2)$$

Furthermore, if frame coords are distorted by an amount $(\Delta u, \Delta v)$, then (M) also predicts equivalent distortions in the sky projection:

$$\Delta x = S_1 \cos(\delta\theta).\Delta u - S_2 \sin(\delta\theta).\Delta v \qquad (3)$$
$$\Delta y = S_1 \sin(\delta\theta).\Delta u + S_2 \cos(\delta\theta).\Delta v \qquad (4).$$

The linear terms of the distortion polynomials $f(u,v)$ and $g(u,v)$: assuming zero skew are (in zero-index notation):

$$\Delta u = A_{10}.u + A_{01}.v \qquad (5)$$
$$\Delta v = B_{10}.u + B_{01}.v \qquad (6).$$

We substitute (5) & (6) into (3) & (4) to yield equivalent linear corrections in the sky plane:

$$\Delta x = \left[S_1 \cos(\delta\theta).A_{10} - S_2 \sin(\delta\theta) B_{10}\right].u +$$

$$\left[S_1 \cos(\delta\theta) A_{01} - S_2 \sin(\delta\theta) B_{01}\right].V \quad (7)$$

$$\Delta y = \left[S_1 \sin(\delta\theta) A_{10} + S_2 \cos(\delta\theta) B_{10}\right].u +$$

$$\left[S_1 \sin(\delta\theta).A_{01} + S_2 \cos(\delta\theta) B_{01}\right].V \quad (8).$$

Now equate the linear coefficients in (7) & (8) with those in (1) & (2) to yield:

(1) ≡ (7) ⇒

$$\Delta S_1 \cos(\delta\theta) = S_1 \cos(\delta\theta).A_{10} - S_2 \sin(\delta\theta) B_{10} \quad (9)$$

$$-\Delta S_2 \sin(\delta\theta) = S_1 \cos(\delta\theta).A_{01} - S_2 \sin(\delta\theta).B_{01} \quad (10)$$

(2) ≡ (8) ⇒

$$\Delta S_1 \sin(\delta\theta) = S_1 \sin(\delta\theta) A_{10} + S_2 \cos(\delta\theta) B_{10} \quad (11)$$

$$\Delta S_2 \cos(\delta\theta) = S_1 \sin(\delta\theta) A_{01} + S_2 \cos(\delta\theta) B_{01} \quad (12)$$

take square of (9) and add it to the square f (11) to yield:

$$\Delta S_1^2 = (S_1 A_{10})^2 + (S_2 B_{10})^2 \quad (13)$$

Similarly, square (10) and (12) and add together:

$$\Delta S_2^2 = (S_1 A_{01})^2 + (S_2 B_{01})^2 \quad (14).$$

By construction, the change in the scales are defined as:

$$\Delta S_1 = S_{1new} - S_{1old}$$

$$\Delta S_2 = S_{2new} - S_{2old}$$

and all the $S_1, S_2$ appearing in the RHS of eqns (13), (14) pertain to the original (old) scales. (13) & (14) can be re-written:

$$\boxed{\begin{aligned} S_{1new} &= S_{1old} + \sqrt{(S_{1old} A_{10})^2 + (S_{2old} B_{10})^2} \qquad (15) \\ S_{2new} &= S_{2old} + \sqrt{(S_{2old} B_{01})^2 + (S_{1old} A_{01})^2} \qquad (16) \end{aligned}}$$

If there were no residual rotation bias (and <u>no</u> skew!), the cross terms: $A_{01} = 0$ and $B_{10} = 0$ and (15), (16) reduce to the familiar refinement relations:

$$S_{1new} = (1 + A_{10}) S_{1old}$$

$$S_{2new} = (1 + B_{01}) S_{2old}$$

To estimate the rotational bias error ($\delta\theta$), we divide eqn (9) by $\cos(\theta)$ to give:

$$\Delta S_1 = S_1 A_{10} - S_2 \tan(\delta\theta) . B_{10} \qquad (17)$$

re-arranging (17), we get:

$$\tan(\delta\theta) = \frac{S_1 A_{10} - \Delta S_1}{S_2 B_{10}} \qquad (18).$$

we substitute eqn. (13) into (18):

$$\Rightarrow \tan(\delta\theta) = \frac{S_1 A_{10} - \sqrt{(S_1 A_{10})^2 + (S_2 B_{10})^2}}{S_2 . B_{10}}$$

with a little algebra, we get:

$$\boxed{\begin{array}{l} \tan(\delta\theta) = \dfrac{-x}{1 + \sqrt{1+x^2}} \\[2ex] \text{where } \quad x = \dfrac{S_2 B_{10}}{S_1 A_{10}} \end{array}} \qquad (19).$$

If the cross-term coefficient $B_{10} = 0$, then it means that $\delta\theta = 0$, as expected.