Wide-field Infrared Survey Explorer

Proposed Raw-Pixel Error Model

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Revision History

Date	Version	Author	Description
March 17, 2008	1.0	Frank Masci	Initial Draft
November 7, 2008	2.0	Frank Masci	Offset in DEB “slope” formula changed from 128 to 1024 by word of mouth from SDL
November 24, 2008	2.1	Frank Masci	* made offsets in DEB “slope” formula explicit for w1-w4 (from recent MIC2 testing); * included LSB truncation factor in DEB formula and propagated this along

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1 Introduction

1.1 Purpose and Scope

We derive a model for the noise variance in observed (down-linked) value for an image pixel. This will be used in the WSDS instrumental calibration pipeline to initiate uncertainties. These will be propagated and updated downstream for every pixel, and stored in image frames to accompany the primary single-frame products. These uncertainties will be used to assess expected versus actual detector noise, and as *prior*-weights to support frame co-addition, source detection and photometry.

1.2 Applicable Documents

- WISE Digital Electronics Box (DEB) processing description (SDL/06-070; Jan 2006): *http://web.ipac.caltech.edu/staff/roc/wise/docs/sdl06-070-.pdf* **(NEWER VERSION KNOWN TO EXIST!)**

- WSDC Functional Requirements Document (WSDC-D-R001; version 2.0; Nov 2007): *http://web.ipac.caltech.edu/staff/roc/wise/docs/WSDC_Functional_Requirements_all.pdf*

- Critical Design Review presentation: Instrumental Calibration (Jan 2008): *http://spider.ipac.caltech.edu/staff/fmasci/home/wise/InstruCal_CDRJan08.pdf*

1.3 Acronyms

ADC	Analog-to-Digital Conversion
ADU	Analog Digital Unit
COV	Covariance
DEB	Digital Electronics Box
DN	Data Number
IPAC	Infrared Processing and Analysis Center
LSB	Least Significant Bit
RN	Read Noise
SDL	Space Dynamics Lab
SNR	Signal to Noise Ratio
SUR	Sample-Up-the-Ramp
VAR	Variance
WISE	Wide-field Infrared Survey Explorer
WSDC	WISE Science Data Center
WSDS	WISE Science Data System

2 Error Model

The observed (down-linked) value m for a pixel is computed on-board from a linear combination of Samples Up the Ramp (SUR) values y_i in DN:

$$m = \frac{1}{2^T}\left[O + \sum_{i=0}^{N} c_i y_i\right], \qquad \text{(Eq. 1)}$$

where nominally $N = 8$, c_i are the SUR coefficients, e.g., {-4,-3,-2,-1,0,1,2,3,4}, O is an offset (as of 11/24/08, $O = 1024$ for all bands), and T is the number of LSB truncations performed on the DEB output (as of 11/24/08, $T = 3, 3, 2, 2$ for bands 1, 2, 3, 4 respectively).

The summation term ($\equiv m2^T - O$) should not be confused with a slope (rate) estimated from a standard linear least-squares fit to the digitized SUR data. This is because the nominal c_i values are not normalized weights for a direct linear filtering/fitting operation. The quantity $m2^T - O$ and the true slope β (in DN / *sample interval*) are related as follows:

$$\beta = \frac{m2^T - O}{K_N}, \quad \text{where} \quad K_N = \sum_{i=0}^{N} i c_i, \qquad \text{(Eq. 2)}$$

In general, the error ε_m for a measured quantity m, which is a function of random variables y_i: $m = f(y_i)$, where $i = 0, 1, 2\ldots N$, can be written by Taylor expanding about its expected value $\langle m \rangle$:

$$\begin{aligned}
\varepsilon_m &= m - \langle m \rangle \\
&\approx \left(y_0 - \langle y_0 \rangle\right)\frac{\partial m}{\partial y_0} + \left(y_1 - \langle y_1 \rangle\right)\frac{\partial m}{\partial y_1} + \ldots + \left(y_N - \langle y_N \rangle\right)\frac{\partial m}{\partial y_N} \\
&\quad + \frac{1}{2!}\left[\left(y_0 - \langle y_0 \rangle\right)^2 \frac{\partial^2 m}{\partial y_0^2} + \left(y_1 - \langle y_1 \rangle\right)^2 \frac{\partial^2 m}{\partial y_1^2} + 2\left(y_0 - \langle y_0 \rangle\right)\left(y_1 - \langle y_1 \rangle\right)\frac{\partial^2 m}{\partial y_0 \partial y_1} + \ldots\right] \\
&\quad + \ldots + \frac{1}{n!}\left[\left(y_0 - \langle y_0 \rangle\right)\frac{\partial}{\partial y_0} + \left(y_1 - \langle y_1 \rangle\right)\frac{\partial}{\partial y_1} + \ldots + \left(y_N - \langle y_N \rangle\right)\frac{\partial}{\partial y_N}\right]^n m,
\end{aligned}$$

where the partial derivatives are evaluated at $\langle y_0 \rangle$, $\langle y_1 \rangle$ … $\langle y_N \rangle$. This is written in this form so that *all* higher order terms that may potentially contribute can be seen. If m is a *linear* combination of the y_i (e.g., Eq. 1), all derivatives 2^{nd} order and above ($n > 1$) vanish. So the total error can be written:

$$\varepsilon_m = \varepsilon_{y_0}\frac{\partial m}{\partial y_0} + \varepsilon_{y_1}\frac{\partial m}{\partial y_1} + \ldots + \varepsilon_{y_N}\frac{\partial m}{\partial y_N}. \qquad \text{(Eq. 3)}$$

The *variance* in m is given by the expectation value of the square of Eq. 3: $\sigma^2_m = \langle \varepsilon_m^2 \rangle$. Squaring and expanding Eq.3, and taking expectation values,

$$\left\langle \varepsilon_m^2 \right\rangle = \left\langle \varepsilon_{y_0}^2 \right\rangle \left(\frac{\partial m}{\partial y_0} \right)^2 + \left\langle \varepsilon_{y_1}^2 \right\rangle \left(\frac{\partial m}{\partial y_1} \right)^2 + \ldots + \left\langle \varepsilon_{y_N}^2 \right\rangle \left(\frac{\partial m}{\partial y_N} \right)^2$$
$$+ 2\left\langle \varepsilon_{y_0} \varepsilon_{y_1} \right\rangle \left(\frac{\partial m}{\partial y_0} \right)\left(\frac{\partial m}{\partial y_1} \right) + 2\left\langle \varepsilon_{y_0} \varepsilon_{y_2} \right\rangle \left(\frac{\partial m}{\partial y_0} \right)\left(\frac{\partial m}{\partial y_2} \right) + \ldots + 2\left\langle \varepsilon_{y_{N-1}} \varepsilon_{y_N} \right\rangle \left(\frac{\partial m}{\partial y_{N-1}} \right)\left(\frac{\partial m}{\partial y_N} \right).$$

Collecting terms, the variance can be written:

$$\sigma_m^2 = \sum_{i=0}^{N} \left(\frac{\partial m}{\partial y_i} \right)^2 \sigma_{y_i}^2 + 2\sum_{i=1}^{N} \sum_{j>i}^{N} \left(\frac{\partial m}{\partial y_i} \right)\left(\frac{\partial m}{\partial y_j} \right) \mathrm{cov}\left(y_i, y_j \right), \qquad \text{(Eq. 4)}$$

where $\mathrm{cov}(y_i , y_j) \equiv <\varepsilon_{yi}\varepsilon_{yj}>$ represents the covariance between any two ramp samples i, j ($i \neq j$). The variance in Eq. 4 can be decomposed into two terms, one containing the *uncorrelated* noise contributions, and the other the *correlated* noise contributions, i.e.,

$$\sigma_m^2 = \mathrm{var}(m) + \mathrm{cov}(m) \qquad \text{(Eq. 5)}$$

We consider each term in turn.

Uncorrelated variance term: var(m)

From Eq. 1 and the first summation term on the right in Eq. 4,

$$\mathrm{var}(m) = \frac{1}{2^{2T}} \sum_{i=0}^{N} c_i^2 \sigma_{y_i}^2 . \qquad \text{(Eq. 6)}$$

Assuming that Poisson (photon) noise and read-noise are the main contributors to the total noise variance σ^2_{yi} in any ramp sample i, we have

$$\sigma_{y_i}^2 = \frac{\Delta y_i}{g} + \frac{\sigma_{RN}^2}{g^2}, \qquad \text{(Eq. 7)}$$

where the first term on the right is the contribution from Poisson noise, i.e., Δy_i is the total count in DN resulting from photoelectrons accumulated from $t = 0$ to sample $t = i$, i.e., $\Delta y_i = y_i - y_0$; g is the gain in *electrons*/DN; and σ_{RN} is the detector read-noise in *electrons per pixel per ramp sample*. This ensures that *sigma* $(= \sqrt{\sigma^2_{yi}})$ will have units of DN. Substituting Eq. 7 into 6, we have:

$$\mathrm{var}(m) = \frac{1}{2^{2T} g} \sum_{i=0}^{N} c_i^2 \Delta y_i + \frac{\sigma_{RN}^2}{2^{2T} g^2} \sum_{i=0}^{N} c_i^2 \qquad \text{(Eq. 8)}$$

Now we use a crucial approximation to simplify Eq. 8. We assume that Δy_i can be estimated from a *linear* rate β:

$$\Delta y_i = y_i - y_0 \approx \beta i, \tag{Eq. 9}$$

where $i = 0, 1, 2 \ldots N$, and y_0 is the y-intercept of the ramp whose explicit value we don't need. A non-zero y_0 may imply residual charge on an array following a reset. Using Eq. 9 in Eq. 1, it can be shown that β takes the same form as Eq. 2 – i.e., we approximate the count rate as the slope from a linear least-squares fit, and this can be scaled from the observed value ($m2^T - O$). Therefore, Eq. 9 can be written:

$$\Delta y_i \approx \left(\frac{m2^T - O}{K_N}\right) i, \tag{Eq. 10}$$

where K_N was defined in Eq. 2. The reason why this is an approximation is because we have neglected non-linear behavior in the ramp. The change in observed DN (or collected photoelectrons) high up the ramp is expected to be smaller than that at the low end (i.e., the linear regime). A straight line fit will provide an average rate over the whole ramp. One can easily extend Eq. 10 to include non-linear terms, i.e., as calibrated from a non-linearity model, however, the curvature in a ramp as measured by its deviation from a linear fit is expected to be less than a few percent for the WISE arrays (from SDL characterization plan). The assumption in Eq. 10 should not significantly impact the Poisson-variance estimation.

Substituting Eq. 10 into Eq. 8, we have the final expression for the *uncorrelated* (first) variance term in Eq. 5:

$$\mathrm{var}(m) \approx \left(\frac{m2^T - O}{2^{2T} g K_N}\right) \sum_{i=0}^{N} i c_i^2 + \left(\frac{\sigma_{RN}^2}{2^{2T} g^2}\right) \sum_{i=0}^{N} c_i^2, \tag{Eq. 11}$$

Correlated variance term: cov(m)

We now simplify the second (double summation) term on the right in Eq. 4. This term *only* accounts for correlated noise between any two samples in the ramp. The correlations are primarily from photon-noise increments in the ramp. Very briefly, a photon-noise fluctuation early on in the ramp will be imprinted onto all subsequent samples because the count at any ramp sample is the cumulative signal up to that point. Therefore, the total error in a sample will *depend* on (or be correlated with) the error from *photon-noise* in any sample lower down.

This should be compared to the effect of read-noise. Read-noise comes into play when the total accumulated charge at a sample is "counted". This is done in a non-destructive manner, where each pixel can be thought as a capacitor. As photons are detected, charge builds up on the capacitor. The counting (or read) process contributes an additional component of noise that is *independent* from one sample to the next. In other words, the value of the accumulated signal *after a read* is not propagated along to affect measurements higher-up on the ramp. It is merely stored.

We are interested in deriving an expression for the covariance between errors in any two samples up the ramp, i.e., cov(y_i , y_j) in Eq. 4. The total error in two ramp samples i, j where $j > i$ can be written:

$$\varepsilon_i = \varepsilon_{ui} + \sum_{k=0}^{i} \Delta\varepsilon_k$$

$$\varepsilon_j = \varepsilon_{uj} + \sum_{l=0}^{j} \Delta\varepsilon_l, \qquad \text{(Eq. 12)}$$

where :

$$\Delta\varepsilon_k = \varepsilon_{k+1} - \varepsilon_k; \quad \Delta\varepsilon_l = \varepsilon_{l+1} - \varepsilon_l$$

are *photon* noise increments and ε_{ui} , ε_{uj} are the *uncorrelated* noise components. We are interested in the expectation value of the product (covariance) between the total error in samples i and j:

$$\mathrm{cov}(y_i, y_j) = \left\langle \varepsilon_i \varepsilon_j \right\rangle = \sum_{k=0}^{i}\sum_{l=0}^{j} \left\langle \Delta\varepsilon_k \Delta\varepsilon_l \right\rangle + \sum_{k=0}^{i} \left\langle \Delta\varepsilon_k \Delta\varepsilon_{uj} \right\rangle + \sum_{l=0}^{j} \left\langle \Delta\varepsilon_l \Delta\varepsilon_{ui} \right\rangle + \left\langle \varepsilon_{ui}\varepsilon_{uj} \right\rangle \qquad \text{(Eq. 13)}$$

The last three terms in Eq. 13 are zero since the ε_{ui} , ε_{uj} components are not correlated with anything except themselves. The photon-noise increments are also independent and are only correlated with themselves, i.e., for instances when $k = l$. Assuming $i < j$, Eq. 13 reduces to:

$$\begin{aligned} \mathrm{cov}(y_i, y_j) &= \sum_{k=0}^{i} \left\langle \Delta\varepsilon_k^2 \right\rangle \\ &= \sum_{k=0}^{i} \sigma^2_{\Delta y_i} \\ &= \sum_{k=0}^{i} \frac{\left[y_{k+1} - y_k \right]}{g} \\ &= \frac{y_i - y_0}{\mathrm{g}} \end{aligned} \qquad \text{(Eq. 14)}$$

where we have assumed that the variance in a signal increment in a SUR interval is dominated by Poisson noise and g is the gain in *electrons*/DN. Under this construction, cov(y_i , y_j) will have units of DN^2. We have therefore shown that the covariance between any two ramp samples i, j (with $i < j$) depends only on the total count accumulated from $t = 0$ up to the sample that is lower on the ramp (i.e., here it's sample i).

Adopting the same formalism as above to approximate the accumulated count $\Delta y_i = y_i - y_0$ in Eq. 14 using a linear count rate (i.e., Eq. 10), the second term on the right in Eq. 4 can be written:

$$\mathrm{cov}(m) \approx 2\left(\frac{m2^T - O}{2^{2T} gK_N} \right) \sum_{i=0}^{N} \sum_{j>i}^{N} i c_i c_j, \qquad \text{(Eq. 15)}$$

where we used Eq. 1 to evaluate the partial derivatives.

In summary, the predicted variance in the actual observed (down-linked) value *m* for a pixel in a WISE array is given by combining Equations 5, 11 and 15:

$$\sigma_m^2 \approx \left(\frac{m2^T - O}{2^{2T} g K_N}\right)\left[\sum_{i=0}^{N} i c_i^2 + 2\sum_{i=0}^{N}\sum_{j>i}^{N} i c_i c_j\right] + \left(\frac{\sigma_{RN}^2}{2^{2T} g^2}\right)\sum_{i=0}^{N} c_i^2, \qquad \text{(Eq. 16)}$$

where:

m, O and T are defined in Eq. 1 and the text below it;

$g =$ gain in electrons/DN in any SUR;

c_i with $i \in \{0, 1, 2, 3 \ldots N\}$ are the predetermined SUR coefficients;

$N =$ maximum number of samples $-$ 1. Nominally $N = 8$;

$K_N = \sum_{i=0}^{N} i c_i$;

$\sigma_{RN} =$ readnoise in electrons/pixel in any SUR.

It's important to note that to compare the expected uncertainty (*sigma*[*m*] = $\sqrt{\sigma^2_m}$) for different sets of SUR coefficients c_i, the coefficients must first be scaled to give $K_N = 1$. This entails dividing the raw coefficients by their value of K_N. For the nominal set of coefficients {-4,-3,-2,-1,0,1,2,3,4}, $K_N = 60$. The value of "$m2^T - O$" and *sigma* can then be computed using these new 'rescaled' c_i values. This will ensure $m2^T - O$ and *sigma* will be in units of DN / *sample interval* and no further scaling factors are involved. I.e., these are the slope (rate) and uncertainty that will result from a direct linear least-squares fit to the SUR data. Note that *no* prior scaling is needed if signal-to-noise ratios "($m2^T - O$) / *sigma*" are being compared instead.

3 Model Validation

3.1 Band 1

The above error model was validated using a Monte Carlo simulation. Parameters for the WISE band-1 detector were used in the first experiment: expected mean background = 8 e^-/sec/pixel; 1-σ read-noise per ramp sample = 19 e^-; gain = 5 e^-/DN (really my own guesses), and the sampling interval in a ramp corresponds to 1.1 sec. We set $T = 0$.

100,000 ramps (independent realizations) were simulated using the above photoelectron rate. Gaussian distributed read and photon noise (containing an implicit *correlated* component) were added to the ramp samples. Statistics were then computed on the whole ensemble. Two separate sets of SUR coefficients were assumed: the nominal {-4,-3,-2,-1,0,1,2,3,4}, and a recently proposed set {0,-7,-5,-3,-1,1,3,5,7} to avoid loss in sensitivity brought about by unreliable/unpredictable behavior in the first ($i = 0$) sample (also known as "first sample effect"). Results are summarized below.

```
Simulation results:
   coeffs: {0,-7,-5,-3,-1,1,3,5,7}
       mean slope (rate) = 1.759744 DN/sample interval.
       sigma = 0.632193 DN/sample interval.
       snr = 2.783556

   coeffs: {-4,-3,-2,-1,0,1,2,3,4}
       mean slope (rate) = 1.760152 DN/sample interval.
       sigma = 0.537267 DN/sample interval.
       snr = 3.276120


Error Model results:
   coeffs: {0,-7,-5,-3,-1,1,3,5,7}
       input slope (true rate) = 1.76 DN/sample interval
       sigma = 0.631099 DN/sample interval
       snr = 2.788786 DN/sample interval

   coeffs: {-4,-3,-2,-1,0,1,2,3,4}
       input slope (true rate) = 1.76 DN/sample interval
       sigma = 0.537267 DN/sample interval
       snr = 3.276120
```

3.2 Band 4

In the second experiment, parameters for the WISE band-4 detector were used: expected mean background = 1200 e^-/sec/pixel; 1-σ read-noise per ramp sample = 55 e^-; gain = 5 e^-/DN (really my own guesses), and the sampling interval in a ramp corresponds to 1.1 sec. We set $T = 0$.

100,000 ramps (independent realizations) were simulated using the above photoelectron rate. Gaussian distributed read and photon noise (containing an implicit *correlated* component) were added to the ramp samples. Statistics were then computed on the whole ensemble. Two separate sets of SUR coefficients were assumed: the nominal {-4,-3,-2,-1,0,1,2,3,4}, and a recently proposed set {0,-7,-5,-3,-1,1,3,5,7} to avoid loss in sensitivity brought about by unreliable/unpredictable behavior in the first ($i = 0$) sample (also known as "first sample effect"). Results are summarized below.

```
Simulation results:
   coeffs: {0,-7,-5,-3,-1,1,3,5,7}
       mean slope (rate) = 263.993495 DN/sample interval.
       sigma = 3.324608 DN/sample interval.
       snr = 79.405882

   coeffs: {-4,-3,-2,-1,0,1,2,3,4}
       mean slope (rate) = 264.000486 DN/sample interval.
       sigma = 3.043115 DN/sample interval.
       snr = 86.753344
```

```
Error Model results:
   coeffs: {0,-7,-5,-3,-1,1,3,5,7}
       input slope (true rate) = 264 DN/sample interval
       sigma = 3.324512 DN/sample interval
       snr = 79.410147 DN/sample interval

   coeffs: {-4,-3,-2,-1,0,1,2,3,4}
       input slope (true rate) = 264 DN/sample interval
       sigma = 3.038530 DN/sample interval
       snr = 86.884108
```

4 Conclusions

1. A model for the variance in observed pixel signal is given by Eq. 16. A future enhancement may include the effects of ramp non-linearity if found to be significant in the context of variance estimation.

2. The error-model predictions for sigma are consistent with those from the simulations. This is encouraging!

3. Assuming doubtful (but the only available) values for the gain and read-noise, the coefficients {0,-7,-5,-3,-1,1,3,5,7} lead to a slightly higher noise in the signal (by ~17 and 9% for bands 1 and 4 respectively) than the nominal coefficients where the first sample is retained. This implies a loss in sensitivity, but is still within requirements when expressed in μJy units. This is a small price to pay, since retention of the first sample (as currently characterized) would lead to a greater loss in sensitivity.

4. The loss in sensitivity when going from the nominal set c_A = {-4,-3,-2,-1,0,1,2,3,4} to c_B = {0,-7,-5,-3,-1,1,3,5,7} (to omit the first sample) also holds true in terms of signal-to-noise ratio (SNR). I.e., for band 1, the SNR drops by ~15% assuming the mean background estimate and simulation parameters in section 3.1. Interestingly, the relative loss $|(SNR_B/SNR_A) - 1|$ from c_A to c_B becomes gradually smaller as the signal (the photon flux) increases. This loss asymptotes to a minimum of ~6% for large signals irrespective of band and all other noise parameters. Therefore, sensitivity losses due to omission of a first sample may only be significant for bands where the mean background signal is low, i.e., bands 1 and 2. For bands 3 and 4 it will be desirable to keep the nominal coefficients c_A if no first-sample effect is present. This assumes each band can have its own SUR coefficients.

5 Acknowledgements

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