PSF Moment Parameters and Uncertainties

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1. Overview

PSFs are computed by averaging background-subtracted volume-normalized point-source postage stamps in a 3×3 oversampling grid. Raw postage stamps are 9×9, so the oversampling grid is 27×27. Averaging is done via a two-pass process with outlier rejection, separately in a 5x5 partition of the array and in one global average. By default, the latter average omits data from pixels whose distance from the array center is more than 91.5% of the distance from the center to a corner. Currently, any postage stamp containing a NaN or masked pixel is discarded.

A given postage stamp may be shifted by ±1 oversampling pixel on each axis to position the point-source centroid over the central oversampling pixel, (14,14), whose central (X,Y) coordinates are (13.5,13.5). Averaging may be done by including each array pixel in all nine projected oversampling pixels beneath it (called "smoothed", because this convolves the oversampling pixel size into the resulting PSF) or only into the one oversampling pixel directly beneath the array pixel center (called "interlaced", the default).

Interlaced mode not only avoids the slight broadening of the PSF but also leaves the noise in the 27×27 grid uncorrelated; it does require a greater data volume, however, than the smoothing mode. Interlaced mode is assumed in the uncertainty calculations in that the pixels are assumed uncorrelated. All uncertainties in the moment parameters stem from the PSF pixel uncertainties, which are computed from the dispersions about the pixel averages divided by $\sqrt{(N\text{-}1)}$, where N is the number of samples averaged in the given pixel.

2. Centroids

$$X_i \equiv i - \tfrac{1}{2} \quad , \quad Y_j \equiv j - \tfrac{1}{2}$$

$$\overline{X} = \sum_{ij} p_{ij} X_i \quad , \overline{Y} = \sum_{ij} p_{ij} Y_j$$

where p_{ij} is the value of pixel (i,j). Note that the volume normalization of the PSF removes the need to divide the summation by the sum of the p_{ij}. We can write p_{ij} as the sum of a true value, indicated by an overparenthesis, and an error, indicated by ε_{ij}. Other errors will also be indicated by ε, with an appropriate subscript.

$$p_{ij} = \widehat{p}_{ij} + \varepsilon_{ij}$$

$$\overline{X} = \widehat{\overline{X}} + \varepsilon_{\overline{X}} \quad , \overline{Y} = \widehat{\overline{Y}} + \varepsilon_{\overline{Y}}$$

Using these in the centroid equations and subtracting off true values from each side results in

$$\varepsilon_{\overline{X}} = \sum_{ij} \varepsilon_{ij} X_i$$

$$\sigma_{ij}^2 = \left\langle \varepsilon_{ij}^2 \right\rangle$$

$$\sigma_{\overline{X}}^2 = \left\langle \varepsilon_{\overline{X}}^2 \right\rangle = \left\langle \left(\sum_{ij} \varepsilon_{ij} X_i \right)^2 \right\rangle \Rightarrow \sum_{ij} \sigma_{ij}^2 X_i^2$$

where cross-pixel terms are dropped in the expectation value of the squared summation because the pixels errors are taken to be uncorrelated. The uncertainty in the Y centroid is given by the corresponding equation:

$$\sigma_{\overline{Y}}^2 = \sum_{ij} \sigma_{ij}^2 Y_j^2$$

3. Principal Axes of the Second Central Moments

The second central moments of the PSF are just the elements of the response covariance matrix:

$$M = \begin{pmatrix} M_X & M_{XY} \\ M_{XY} & M_Y \end{pmatrix}$$

$$M_X = \sum_{ij} (X_i - \overline{X})^2 p_{ij}$$

$$M_Y = \sum_{ij} (Y_j - \overline{Y})^2 p_{ij}$$

$$M_{XY} = \sum_{ij} (X_i - \overline{X})(Y_j - \overline{Y}) p_{ij}$$

Like the centroids, these elements are linear in p_{ij}, and a similar "true plus error" formulation leads to

$$V_X = \sum_{ij} \left(X_i - \overline{X} \right)^2 \sigma_{ij}^2$$

$$V_Y = \sum_{ij} \left(Y_j - \overline{Y} \right)^2 \sigma_{ij}^2$$

$$V_{XY} = \sum_{ij} \left(X_i - \overline{X} \right)\left(Y_j - \overline{Y} \right) \sigma_{ij}^2$$

where we use V to indicate variance rather than σ^2 in order to accommodate the covariance element (i.e., to avoid the appearance of a squared quantity for something that may be negative). Finding the principal axes involves diagonalizing M; this is straightforward for a 2-D real matrix:

$$A \equiv M_X + M_Y$$

$$B \equiv \sqrt{(M_X - M_Y)^2 + 4\,M_{XY}^2}$$

$$S_{maj} = \sqrt{\frac{A+B}{2}} \quad , \quad S_{min} = \sqrt{\frac{A-B}{2}}$$

The angle from the X axis to the major principal axis is undefined if the principal axes are equal, i.e., if the PSF is circular, in which case, the off-diagonal element will be zero. It may also happen that the principal axes are not equal, but the off-diagonal element is nevertheless zero, in which case the PSF is already aligned with the X and Y axes. Taking these possibilities into account:

$$
\begin{array}{l}
\text{if } \left|M_{XY}\right| < 10^{-25}\, M_{maj} \text{ then} \\
\quad \gamma = 0^\circ \\
\quad \text{if } M_Y > M_X \quad , \quad \gamma = 90^\circ \\
\text{else} \\
\quad \gamma = \tan^{-1}\left(\dfrac{M_X - M_{maj}}{-M_{XY}} \right) \\
\text{end if}
\end{array}
$$

The uncertainties of the principal axes are obtained as follows, where some care is needed to avoid division by zero.

$$
\begin{array}{l}
Q \equiv (M_X - M_Y)^2 + 4 M_{XY}^2 \\
\text{if } Q > 0 \text{ then} \\
\quad D = \sqrt{Q} \\
\quad T = \dfrac{M_X - M_Y}{D} \\
\text{else} \\
\quad D = 9.9 \times 10^{99} \\
\quad T = 0 \\
\text{end if}
\end{array}
$$

These temporary variables are needed in evaluating the derivatives of the principal moments with respect to the elements of *M*:

$$\frac{\partial S_{maj}}{\partial M_X} = \frac{1}{4} + \frac{T}{2} \quad , \quad \frac{\partial S_{min}}{\partial M_X} = \frac{1}{4} - \frac{T}{2}$$

$$\frac{\partial S_{maj}}{\partial M_Y} = \frac{1}{4} - \frac{T}{2} \quad , \quad \frac{\partial S_{min}}{\partial M_Y} = \frac{1}{4} + \frac{T}{2}$$

$$\frac{\partial S_{maj}}{\partial M_{XY}} = \frac{M_{XY}}{D} \quad , \quad \frac{\partial S_{min}}{\partial M_{XY}} = \frac{-M_{XY}}{D}$$

The errors will be assumed small enough to allow a Taylor's expansion with terms of second and higher order dropped. For example, the uncertainty variance in the semimajor axis S_{maj} is:

$$\varepsilon_{S_{maj}} = \frac{\partial S_{maj}}{\partial M_X} \varepsilon_{M_X} + \frac{\partial S_{maj}}{\partial M_Y} \varepsilon_{M_Y} + \frac{\partial S_{maj}}{\partial M_{XY}} \varepsilon_{M_{XY}}$$

Squaring and taking expectation values gives:

$$\sigma^2_{S_{maj}} = \left(\frac{\partial S_{maj}}{\partial M_X} \right)^2 V_X + \left(\frac{\partial S_{maj}}{\partial M_Y} \right)^2 V_Y + \left(\frac{\partial S_{maj}}{\partial M_{XY}} \right)^2 V_{XY}$$

Similar equations apply for the uncertainties in S_{min} and γ. The derivatives for γ involve lengthier expressions and are omitted here; they will be supplied upon request.

4. Axis Ratio

The ratio of the minor principal axis to the major principal axis is $R = S_{min}/S_{maj}$. The uncertainty is computed as follows.

$$\begin{aligned} \sigma^2_R &= \left(\frac{\partial R}{\partial S_{maj}} \right)^2 \sigma^2_{S_{maj}} + \left(\frac{\partial R}{\partial S_{min}} \right)^2 \sigma^2_{S_{min}} \\ &= \left(\frac{-S_{min}}{S^2_{maj}} \right)^2 \sigma^2_{S_{maj}} + \frac{\sigma^2_{S_{min}}}{S^2_{maj}} \end{aligned}$$

5. Noise Pixel

The noise pixel value is computed as follows, where the summations are limited to a radial distance from the centroid R_{eff}.

$$S_1 = \sum_{ij:\,X_i^2+Y_j^2 \le R_{eff}^2} p_{ij}$$

$$S_2 = \sum_{ij:\,X_i^2+Y_j^2 \le R_{eff}^2} p_{ij}^2$$

$$N_P = \frac{S_1^2}{9\,S_2}$$

where the 9 in the denominator scales the value to the array-pixel size, i.e., accounts for the 3×3 oversampling. The uncertainty can be obtained by brute force as follows, where the summation notation is simplified but must be understood as being taken within the R_{eff} region.

$$\frac{\partial N_P}{\partial p_{ij}} = \frac{2S_1}{9S_2} - \frac{2S_1^2\,p_{ij}}{9S_2^2}$$

$$\sigma_{N_P}^2 = \sum_{ij} \left(\frac{\partial N_P}{\partial p_{ij}} \right)^2 \sigma_{ij}^2$$

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